FOURIER TRANSFORM - FINITE WAVE TRAIN

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 26 June 2025.

The Fourier transform of a function $F(t)$ is given by

$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(t)\,e^{-i\omega t}dt \tag{1}$$

Here we present an example of a finite wave train, which is a function that is a sine wave over a restricted domain, and zero elsewhere.

Find the Fourier transform of

$$F(t)=\frac{\sin\pi t}{1-t^2}=-\frac{\sin\pi t}{t^2-1} \tag{2}$$

Writing the sine as

$$\sin\pi t=\frac{e^{i\pi t}-e^{-i\pi t}}{2i} \tag{3}$$

we have from 1

$$G(\omega)=-\frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{e^{it(\pi-\omega)}-e^{-it(\pi+\omega)}}{(t-1)(t+1)}dt \tag{4}$$

The integral in 4 can be done by using indented contours around the poles at $t=\pm1$. Consider the first exponential:

$$G_1(\omega)=-\frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{e^{it(\pi-\omega)}}{(t-1)(t+1)}dt \tag{5}$$

We would like to use a contour consisting of a large semicircle in either the upper or lower half plane, with small semicircular arcs around $t=\pm1$. To have the integral around the large semicircle go to zero, we can invoke Jordan's lemma, which is valid if $\pi-\omega>0$. In this case, the remaining portions of the contour integral are

$$-\frac{1}{4\pi i}\left[\int_{-\infty}^{-s}+\int_{S}+\int_{s}^{-r}+\int_{R}+\int_{r}^{\infty}\right]\frac{e^{it(\pi-\omega)}}{(t-1)(t+1)}dt \tag{6}$$

where $\int_S$ is the integral around the indentation at $t=-1$, $\int_s^{-r}$ is the integral along the path between the two indentations, and $\int_R$ is the integral around the indentation at $t=+1$. In the limit as $s\to0$ and $r\to0$ We have

$$\int_{-\infty}^{\infty}\frac{e^{it(\pi-\omega)}}{(t-1)(t+1)}dt=-\left[\int_{S}+\int_{R}\right] \tag{7}$$

We use the formula for evaluating integrals around indentations where the indentation is a clockwise contour spanning a half circle, so that the angular range is $-\pi$. We have

$$\left[\int_{S}+\int_{R}\right]\frac{e^{it(\pi-\omega)}}{(t-1)(t+1)}dt=-i\pi\left[\mathrm{Res}(-1)+\mathrm{Res}(1)\right] \tag{8}$$

The residues are

$$\mathrm{Res}(-1)=\frac{e^{-i(\pi-\omega)}}{-2} \tag{9}$$

$$\mathrm{Res}(1)=\frac{e^{i(\pi-\omega)}}{2} \tag{10}$$

so we have

$$-\left[\int_{S}+\int_{R}\right]\frac{e^{it(\pi-\omega)}}{(t-1)(t+1)}dt=i\pi\left(\frac{e^{i(\pi-\omega)}}{2}-\frac{e^{-i(\pi-\omega)}}{2}\right) \tag{11}$$

$$=-\pi\sin(\pi-\omega) \tag{12}$$

$$=-\pi\sin\omega \tag{13}$$

Therefore

$$-\frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{e^{it(\pi-\omega)}}{(t-1)(t+1)}dt=-\frac{1}{4\pi i}(-\pi\sin\omega)=-\frac{i}{4}\sin\omega;\quad\omega<\pi \tag{14}$$

When $\omega>\pi$, we can use a contour with a large semicircle in the lower half plane, again with indentations around $t=\pm1$. This time, however, the indentations extend into the lower half plane, so the contour is counterclockwise rather than clockwise as it was for the case $\omega<\pi$. This flips the sign in the $\int_S+\int_R$ integrals, but leaves everything else (including the residues) unchanged. Thus we get

$$-\frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{e^{it(\pi-\omega)}}{(t-1)(t+1)}dt=\frac{i}{4}\sin\omega;\quad\omega>\pi \tag{15}$$

For the second term in 4, the process is the same. For $\omega<-\pi$ we use the upper contour with the result

$$-\left[\int_{S}+\int_{R}\right]\frac{e^{-it(\pi+\omega)}}{(t-1)(t+1)}dt=i\pi\left[\mathrm{Res}(-1)+\mathrm{Res}(1)\right] \tag{16}$$

$$= i\pi\left(\frac{e^{i(\pi+\omega)}}{-2}+\frac{e^{-i(\pi+\omega)}}{2}\right) \tag{17}$$

$$= \pi\sin(\pi+\omega) \tag{18}$$

$$= -\pi\sin\omega \tag{19}$$

Therefore

$$-\frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{e^{-it(\pi+\omega)}}{(t-1)(t+1)}dt = -\frac{1}{4\pi i}(-\pi\sin\omega) = -\frac{i}{4}\sin\omega; \quad \omega<-\pi \tag{20}$$

For $\omega>-\pi$, again the only change is flipping the sign of the result, so we have

$$-\frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{e^{-it(\pi+\omega)}}{(t-1)(t+1)}dt = \frac{i}{4}\sin\omega; \quad \omega>-\pi \tag{21}$$

Combining 14, 15, 20 and 21 to get 4 we have

$$G(\omega) = \begin{cases} -\frac{i}{4}\sin\omega-\left(-\frac{i}{4}\sin\omega\right)=0 & \omega<-\pi \\ -\frac{i}{4}\sin\omega-\frac{i}{4}\sin\omega=-\frac{i}{2}\sin\omega & -\pi<\omega<\pi \\ \frac{i}{4}\sin\omega-\frac{i}{4}\sin\omega=0 & \omega>\pi \end{cases} \tag{22}$$

Thus we have a single cycle of a sine wave between $\omega=-\pi$ and $\omega=\pi$, with the function being zero outside these limits. This is a *finite wave train*. The function G is continuous everywhere, since the endpoints of the three segments match up. A plot (apart from the imaginary factor i) is shown in Fig. 1.

We can combine all this into a single formula by using the Heaviside step function $H(x)$, defined as

$$H(x) = \begin{cases} 0 & x<0 \\ 1 & x\ge 0 \end{cases} \tag{23}$$

We get

$$G(\omega) = \frac{i}{2}\sin\omega\left(H(\omega-\pi)-H(\omega+\pi)\right) \tag{24}$$

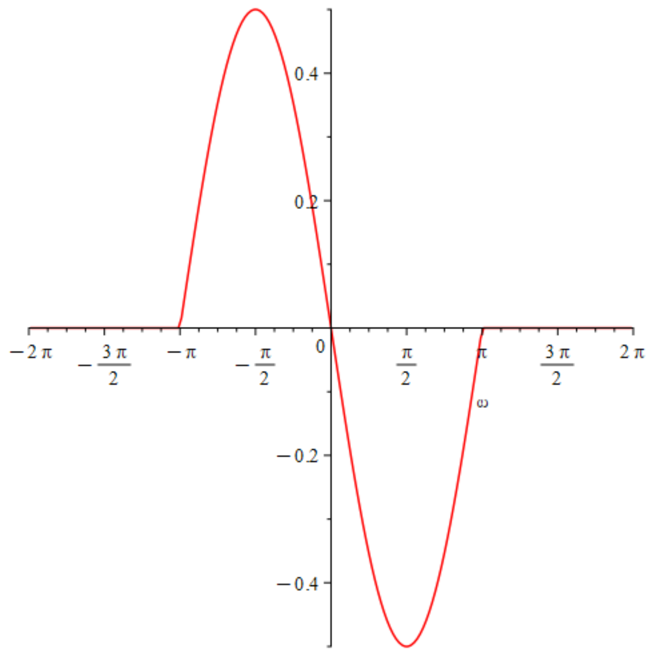

FIGURE 1. The transform $G(\omega)$.